Algebra 5.3

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Algebra 5.3 has $p^4+5p^3+19p^2+64p+140+(p+6)\gcd(p-1,3)+(p+7)\gcd(p-1,4)+\gcd(p-1,5)$ immediate descendants of order p^7 and p-class 3.

Algebra 5.3 has presentation

$$\langle a,b,c,d \mid ca, da, cb, db, dc, pa, pb, pc, pd, \text{ class } 2\rangle.$$

So it has characteristic p and derived algebra of order p generated by ba, with all other commutators trivial. So if L is an immediate descendant of 5.3 then L has class 3, L_3 is generated by baa, bab, and the elements $ca, da, cb, db, dc, pa, pb, pc, pd$ are all linear combinations of baa, bab. The commutator structure of L must correspond to one of the algebras 7.21 – 7.28 in the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of order p^7. So we can assume that one of the following sets of commutator relations holds. For any given set of commutator relations, pa, pb, pc, pd are linear combinations of baa, bab.

$$\begin{aligned}
ca &= cb=da=db=dc=0,\\
cb &= da=db=dc=0,\ ca=bab,\\
cb &= da=db=dc=0,\ ca=baa,\\
da &= db=dc=0,\ ca=bab,\ cb=\omega baa,\\
ca &= da=dc=0,\ cb=baa,\ db=bab,\\
da &= dc=0,\ ca=db=bab,\ cb=baa,\\
da &= dc=0,\ ca=db=bab,\ cb=\omega baa,\\
ca &= cb=da=db=0,\ dc=baa,\\
cb &= da=db=0,\ ca=bab,\ dc=baa.
\end{aligned}$$

In 6 of these cases we are able to provide parametrized presentations with fairly simple restrictions on the parameters, but in cases 4, 6 and 7 we were unable to do this.

1 Case 4

We are able to provide parametrized presentations with fairly simple restrictions on the parameters in Case 4 for, except for one presentation

$\langle a, b, c, d \,|\, ca{-}bab,\ cb{-}\omega baa, da, db, dc,\ pa{-}\lambda baa{-}\mu bab,\ pb{-}\nu baa{-}\xi bab,\ pc, pd, \text{ class } 3\rangle.$

If we write the parameters λ, μ, ν, ξ in a matrix (which is assumed to be non-singular)

$$A = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix},$$

then two matrices give isomorphic algebras if and only if they are in the same orbit under the action

$$A \to \frac{1}{\det P} PAP^{-1},$$

where P lies in the group of non-singular matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix}.$$

This is the same action as appears in algebra 6.178 in the algebras of order p^6. In fact the subalgebra $\langle a, b, c\rangle$ here is 6.178. There is a Magma program to compute the orbits in notes5.3.m. In the notes on 6.178 I commented that it would be nice to do better than complexity p^5 in the program to sort the orbits, and I see that the program here has complexity p^4.

2 Case 6

In Case 6, L satisfies the commutator relations $da = dc = 0$, $ca = db = bab$, $cb = baa$. We write

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

where A is 4×2 matrix. Two matrices A give isomorphic algebras if they lie in the same orbit under the action

$$A \to \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \pm\beta & \pm\alpha & \pm\lambda & \pm\mu \\ 0 & 0 & \alpha^2 - \beta^2 & -4\alpha\beta \\ 0 & 0 & \pm\alpha\beta & \pm(\alpha^2 - \beta^2) \end{pmatrix} A \begin{pmatrix} \pm\alpha & \mp\beta \\ \beta & \alpha \end{pmatrix}^{-1}.$$

There is a Magma program to compute the orbits in notes5.3.m.

3 Case 7

In Case 7, L satisfies the commutator relations $da = dc = 0$, $ca = db = bab$, $cb = \omega baa$. We write

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

where A is 4×2 matrix. Two matrices A give isomorphic algebras if they lie in the same orbit under the action

$$A \to \frac{1}{\alpha^2 + \omega\beta^2} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \mp\omega\beta & \pm\alpha & \pm\lambda & \pm\mu \\ 0 & 0 & \alpha^2 - \omega\beta^2 & 4\omega\alpha\beta \\ 0 & 0 & \mp\alpha\beta & \pm(\alpha^2 - \omega\beta^2) \end{pmatrix} A \begin{pmatrix} \pm\alpha & \pm\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}.$$

There is a MAGMA program to compute the orbits in notes5.3.m.