Algebra 4.1

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The number of immediate descendants of algebra 4.1 of order p^7 is 1361 if $p = 3$. For $p > 3$ it is $p^5 + 2p^4 + 7p^3 + 25p^2 + 88p + 270 + (p + 4)\gcd(p - 1, 3) + \gcd(p - 1, 4)$.

If L is an immediate descendant of 4.1 of order p^7 then L is generated by a, b, c, d, L_2 has order p^3, and $L_3 = \{0\}$.

1 L abelian

$$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pd, \text{ class } 2\rangle.$$

2 L^2 has order p

If L^2 has order p then we can assume that L^2 is generated by ba and that one of the following two sets of commutator relations hold:

$$\begin{aligned} ca &= da = cb = db = dc = 0, \\ ca &= da = cb = db = 0,\ dc = ba. \end{aligned}$$

There are 7 algebras in the first case, and 4 in the second case.

3 L^2 has order p^2

If L^2 has order p^2 then we can assume that one of the following sets of commutator relations holds:

$$\begin{aligned} da &= cb = db = dc = 0, \\ ca &= da = cb = db = 0, \\ da &= cb = dc = 0,\ db = ca, \\ da &= cb = 0,\ db = ca,\ dc = \omega ba. \end{aligned}$$

Note that L^2 is generated by ba, ca in all but the second of these algebras. In the second algebra, L^2 is generated by ba, dc. We obtain $2p+29$ algebras in the first case, $(p^2-1)/2+4p+30$ in the second, $3p+26$ in the third, and $(p^2-1)/2+2p+6$ in the fourth.

In solving the isomorphism problem in Case 4, we have the following presentation:

$$\langle a, b, c, d \mid da, cb, db - ca, dc - \omega ba, pa, pb - xba - yca, pc - zba - tca, \text{ class } 2\rangle,$$

where $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ runs over a set of representatives for the equivalence classes of non-singular matrices A under the equivalence relation given by

$$A \sim \alpha^{-1} \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix} A \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix}^{-1}.$$

There are $(p+1)^2/2$ equivalence classes.

There is a Magma program in notes4.1case4.m to compute a set of representative matrices A.

4 L^2 has order p^3

If L^2 has order p^3 then L must have the same commutator structure as one of 7.15 – 7.20 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$, so we can assume that one of the following sets of commutator relations holds:

$$\begin{aligned} da &= db = dc = 0, \\ ca &= da = db = 0, \\ ca &= da = dc = 0, \\ ca &= da = 0,\ dc = ba, \\ da &= 0,\ db = ca,\ dc = cb, \\ da &= 0,\ db = \omega ca,\ dc = ba. \end{aligned}$$

In Case 1 we have $3p + 18$ algebras.
In Case 2 we have $\frac{77}{2}p + \frac{173}{2} + 11p^2 + \frac{5}{2}p^3 + \frac{1}{2}p^4$ algebras, but you need to add 2 if $p = 1 \bmod 3$.
In Case 3 we have $p^2 + 3p + 15$, but again you need to add 2 if $p = 1 \bmod 3$.
In Case 4 we have $3p^2 + 13p + 31$ algebras, but we need to add 2 if $p = 1 \bmod 4$ and add 2 if $p = 1 \bmod 3$.
In Case 5 we have 550 algebras when $p = 3$ and

$$\begin{aligned} p^5 + p^4 + 4p^3 + 6p^2 + 18p + 19 \text{ if } p &= 1 \bmod 3, \\ p^5 + p^4 + 4p^3 + 6p^2 + 16p + 17 \text{ if } p &= 2 \bmod 3. \end{aligned}$$

In Case 6 we have $\frac{9}{2}p + \frac{13}{2} + 3p^2 + \frac{1}{2}p^4 + \frac{1}{2}p^3$ algebras.
We need computer programs to sort out the isomorphism problem in Case 5 and in Case 6.

4.1 Case 5

Let L satisfy $da = 0$, $db = ca$, $dc = cb$. It is convenient to replace b by $b+d$, so that L satisfies $da = cb = 0$, $db = ca$. So L^2 is generated by ba, ca and dc, and $pL \leq L^2$. It is fairly easy to see that if a', b', c', d' generate L and satisfy $d'a' = c'b' = 0$, $d'b' = c'a'$, then (modulo L^2)

$$\begin{aligned} a' &= \alpha\lambda a + \beta\lambda b + \beta\mu c - \alpha\mu d, \\ b' &= \gamma\lambda a + \delta\lambda b + \delta\mu c - \gamma\mu d, \\ c' &= \gamma\nu a + \delta\nu b + \delta\xi c - \gamma\xi d, \\ d' &= -\alpha\nu a - \beta\nu b - \beta\xi c + \alpha\xi d \end{aligned}$$

with (α, β) and (γ, δ) linearly independant, and with (λ, μ) and (ν, ξ) linearly independant. Furthermore

$$\begin{pmatrix} b'a' \\ c'a' \\ d'c' \end{pmatrix} = (\alpha\delta - \beta\gamma) \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ dc \end{pmatrix}.$$

So we consider orbits of 4×3 matrices A (representing pa, pb, pc, pd) under transformations of the form

$$A \longmapsto (\alpha\delta - \beta\gamma)^{-1} \begin{pmatrix} \alpha\lambda & \beta\lambda & \beta\mu & -\alpha\mu \\ \gamma\lambda & \delta\lambda & \delta\mu & -\gamma\mu \\ \gamma\nu & \delta\nu & \delta\xi & -\gamma\xi \\ -\alpha\nu & -\beta\nu & -\beta\xi & \alpha\xi \end{pmatrix} A \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}^{-1}.$$

We note that if we multiply $\alpha, \beta, \gamma, \delta$ through by a factor k (in the expression above), and multiply λ, μ, ν, ξ through by a factor l, then the image of A is multiplied by a factor $k^{-1}l^{-1}$. So we can ignore the factor $(\alpha\delta - \beta\gamma)^{-1}$ and still get the same orbits.

We actually have an action of $\mathrm{GL}(2,p) \times \mathrm{GL}(2,p)$ on the vector space of 4×3 matrices, and if we leave out the factor $(\alpha\delta - \beta\gamma)^{-1}$ (as described above) then the kernel of the action is the subgroup $\{(kI, kI) \mid k \neq 0\}$, so (in effect) we have a group of order $p^2 (p-1)^3 (p+1)^2$ acting on a space of order p^{12}.

If we take $\mu = 0$ in the matrices above, then we obtain a subgroup H of the automorphism group of index $p+1$. There are

$$f(p) = p^6 + 2p^5 + 4p^4 + 8p^3 + 15p^2 + 29p + 27 + (2p + 3)\gcd(p - 1, 3)$$

orbits of matrices under the action of H, and we can "write down" a set of representatives for these orbits. However for $p = 19$ this takes about 3 minutes on my 5 year old linux box, and the representatives take up 4.5 gigabytes of space. So I save space by not writing all the representatives down in the program to generate orbit representatives under the action of the full group G.

There is a Magma program to compute a set of orbit representatives under the action of the full group G in notes4.1case5.m. The representatives are stored as 4×3 matrices over $\mathrm{GF}(p)$, which takes up less space than storing them as integer sequences. We compute a tranversal for the subgroup H in G, and for each of the $f(p)$ H-orbit representatives A, we compute the images of A under elements of the transversal, and determine how the H-orbits fuse under the action of G.Thus we have to consider $(p+1)f(p)$ matrices At where A is an H-orbit representative and t is an element of the transversal. For each such matrix At we compute the H-orbit representative of At. (This takes a bounded amount of work involving arithmetic over $\mathrm{GF}(p)$.) We index the H-orbits, and we add an H-orbit representative A to the list of the G-orbit representatives if the index of the H-orbit containing A is greater than or equal to the indexes of the H-orbits containing the matrices At for t in the transversal. So, if the index of the H-orbit containing At is less than the index of the H-orbit containing A, then we discard A and there is no need to consider the elements Au for u in the remainder of the transversal. This means that we don't actually have to consider all the elements At. For $p = 3$ we only need to consider less than two thirds of the elements At, for $p = 5$ less than a half, for $p = 7$ a little over a third, and so on. Experimentally, it seems that the proportion drops as the prime increases. So the total amount of work needed to compute a set of representatives for the G-orbits is of order somewhere between p^6 and p^7. For $p \leq 23$ the time taken for the program to run is roughly proportional to $p^{6.2}$. However this is a serious bottleneck, and it takes about two hours to generate the list for $p = 19$ on my five year old linux box. Note however that $19^5 = 2476\,099$, and there is probably only a limited amount of interesting work you can do with two and half million groups of order 19^7.

4.2 Case 6

Let L satisfy $da = 0$, $db = \omega ca$, $dc = ba$. Then L^2 is generated by ba, ca, cb and $pL \leq L^2$. It is straightforward to show that all elements in the linear span of a, b, c, d have breadth 3, except for those of the form $\alpha a + \delta d$. Using this we can show that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then (modulo L^2)

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We let

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

where A is a 4×3 matrix over $\mathbb{Z}_p$. Then under a change of generating set of the form described above we see that

$$A \mapsto \begin{pmatrix} \alpha & 0 & 0 & \delta \\ \pm\lambda & \pm\gamma & \pm\omega\beta & \pm\mu \\ \nu & \beta & \gamma & \xi \\ \pm\omega\delta & 0 & 0 & \pm\alpha \end{pmatrix} AB^{-1},$$

where

$$B=\begin{pmatrix} \pm(\alpha\gamma-\omega\beta\delta) & \pm(\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ \pm(\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & \pm(\gamma\lambda-\omega\beta\mu+\omega\gamma\xi-\omega\beta\nu) & \pm(\gamma^2-\omega\beta^2) \end{pmatrix}.$$

We note that $\langle a,d\rangle+L^2$ is a characteristic subalgebra, and first investigate the orbits of pa,pd. We consider three separate cases: $pa=pd=0$, pa and pd span a one dimensional subspace, and pa,pd are linearly independent. It turns out that there are $p+4$ orbits of pa,pd. It is quite easy to see that if pa,pd do *not* span $\langle ba,ca\rangle$ then we can assume that $pa=pd=0$, or $pa=0$, $pd=ca$, or $pa=0$, $pd=cb$, or $pa=ca$, $pd=cb$. There are p orbits where pa,pd span $\langle ba,ca\rangle$, and we have a MAGMA program to find them.

4.2.1 $pa=pd=0$

If pb,pc don't both lie in $\langle ba,ca\rangle$ then we can take $pb\in\langle ba,ca\rangle$ and $pc\notin\langle ba,ca\rangle$, which mean we need to take $\beta=0$. We can then take $pc=cb$, which means we need to take $\gamma=1$ in the $+$ matrices and $\gamma=-1$ in the $-$ matrices. We can then take $pc=0$ or ca. There are p orbits when $pb,pc\in\langle ba,ca\rangle$, and there is a MAGMA program to find them.

4.2.2 $pa=0$, $pd=ca$

We need $\delta=0$, $\beta=0$ in both the plus and minus matrices, and $\gamma=1$ in the plus matrices and $\gamma=-1$ in the minus matrices. We then have:

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ x & y & z \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ (\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & (\gamma\lambda-\omega\beta\mu+\omega\gamma\xi-\omega\beta\nu) & (\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\alpha}(u+w\mu+w\nu) & \frac{1}{\alpha}(v+\mu-w\lambda-w\xi\omega) & w \\ \frac{1}{\alpha}(x+z\mu+z\nu) & \frac{1}{\alpha}(y+\xi-z\lambda-z\xi\omega) & z \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ x & y & z \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ -(\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & -(\gamma\lambda-\omega\beta\mu+\omega\gamma\xi-\omega\beta\nu) & -(\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\alpha}(w\mu-u+w\nu) & -\frac{1}{\alpha}(v-\mu+w\lambda+w\xi\omega) & -w \\ \frac{1}{\alpha}(z\mu-x+z\nu) & \frac{1}{\alpha}(y-\xi+z\lambda+z\xi\omega) & z \\ 0 & 1 & 0 \end{pmatrix}$$

We can assume that $0\le w\le(p-1)/2$. If $w\neq0$ we can assume that $u=v=y=0$, that $x=0$ or 1, with no restriction on z.

If $w=0$ and $z\neq0$, we can assume that $u=0$ or 1, and that $v=x=y=0$.

If $w=z=0$, we can assume that $v=y=0$, and that $u=0$ and $x=0$ or 1, or that $u=1$ and $0\le x\le(p-1)/2$.

4.2.3 $pa=0$, $pd=cb$

We need $\delta=0$, $\lambda=-\xi\omega$, $\mu=-\nu$, $\alpha=\gamma^2-\beta^2\omega$ in both plus and minus matrices, giving:

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ x & y & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ (\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & (\gamma\lambda-\omega\beta\mu+\omega\gamma\xi-\omega\beta\nu) & (\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$
= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(u\gamma^2 - v\beta\gamma - y\beta^2\omega + x\beta\gamma\omega\right) & \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(v\gamma^2 - x\beta^2\omega^2 - u\beta\gamma\omega + y\beta\gamma\omega\right) & \frac{1}{\gamma^2-\beta^2\omega}\left(w\gamma - \nu + z\beta\omega\right) \\ -\frac{1}{(\gamma^2-\beta^2\omega)^2}\left(v\beta^2 - x\gamma^2 - u\beta\gamma + y\beta\gamma\right) & \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(y\gamma^2 + v\beta\gamma - u\beta^2\omega - x\beta\gamma\omega\right) & \frac{1}{\gamma^2-\beta^2\omega}\left(\xi + w\beta + z\gamma\right) \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ x & y & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma - \omega\beta\delta) & -(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ -(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & -(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & -(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(u\gamma^2 - v\beta\gamma - y\beta^2\omega + x\beta\gamma\omega\right) & -\frac{1}{(\gamma^2-\beta^2\omega)^2}\left(v\gamma^2 - x\beta^2\omega^2 - u\beta\gamma\omega + y\beta\gamma\omega\right) & \frac{1}{\gamma^2-\beta^2\omega}\left(w\gamma - \nu + z\beta\omega\right) \\ \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(v\beta^2 - x\gamma^2 - u\beta\gamma + y\beta\gamma\right) & \frac{1}{(\gamma^2-\beta^2\omega)^2}\left(y\gamma^2 + v\beta\gamma - u\beta^2\omega - x\beta\gamma\omega\right) & -\frac{1}{\gamma^2-\beta^2\omega}\left(\xi + w\beta + z\gamma\right) \\ 0 & 0 & 1 \end{pmatrix}
$$

So we can take $w = z = 0$, and we can assume that $u = 0, 1$, or the least non-square. (Experimentally only 0 and 1 arise, but I don't have a proof of this.) There is a Magma program to find the orbits of u, v, x, y.

4.2.4 $pa = ca$, $pd = cb$

We need $\delta = 0$, $\beta = 0$ and $\gamma = 1$ in both the plus and minus matrices. You also need $\lambda = -\xi\omega$, $\mu = -\nu$, and $\alpha = 1$. We then have:

$$
\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ x & y & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (\alpha\gamma - \omega\beta\delta) & (\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ (\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & (\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & (\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix} 0 & 1 & 0 \\ u & v - \xi\omega & w - \nu \\ x & y + \nu & z + \xi \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ x & y & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma - \omega\beta\delta) & -(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ -(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & -(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & -(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix} 0 & 1 & 0 \\ u & \xi\omega - v & w - \nu \\ -x & y + \nu & -z - \xi \\ 0 & 0 & 1 \end{pmatrix}
$$

So you can take $v = w = 0$ and $0 \le x \le (p-1)/2$. If $x = 0$ you can take $0 \le z \le (p-1)/2$.

4.2.5 pa, pd span $\langle ba, ca\rangle$

If pb, pc both lie in $\langle ba, ca\rangle$, then we can assume that $pb = pc = 0$, and that $pa = ca$. There is a Magma program to find the p orbits of pd.

If pb, pc don't both lie in $\langle ba, ca\rangle$, then we can assume that $pb = 0$, and that $pc \in \langle ba, ca\rangle + cb$ though we then need $\beta = 0$, and $\gamma = 1$ in the plus matrices and $\gamma = -1$ in the minus matrices. This gives:

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ x & y & 1 \\ p & q & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma - \omega\beta\delta) & (\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ (\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & (\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & (\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2-\delta^2\omega}\left(q\delta^2 + u\alpha^2 + p\alpha\delta + v\alpha\delta\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(v\alpha^2 + q\alpha\delta + p\delta^2\omega + u\alpha\delta\omega\right) \\ \frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha\mu + q\mu\delta + u\alpha\lambda + v\lambda\delta\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha\mu + v\alpha\lambda + p\mu\delta\omega + u\lambda\delta\omega\right) \\ \frac{1}{\alpha^2-\delta^2\omega}\left(x\alpha + y\delta + \alpha\mu + \alpha\nu - \lambda\delta + p\alpha\xi + q\delta\xi + u\alpha\nu + v\delta\nu - \delta\xi\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}(y\alpha - \alpha\lambda + q\alpha\xi + v\alpha\nu + x\delta\omega - \alpha\xi\omega + \mu\delta\omega + \delta\nu\omega \\ \frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha^2 + q\alpha\delta + v\delta^2\omega + u\alpha\delta\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha^2 + u\delta^2\omega^2 + p\alpha\delta\omega + v\alpha\delta\omega\right) \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ x & y & 1 \\ p & q & 0 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma - \omega\beta\delta) & -(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ -(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & -(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & -(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{\alpha^2-\delta^2\omega}\left(q\delta^2 + u\alpha^2 + p\alpha\delta + v\alpha\delta\right) & -\frac{1}{\alpha^2-\delta^2\omega}\left(v\alpha^2 + q\alpha\delta + p\delta^2\omega + u\alpha\delta\omega\right) \\ -\frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha\mu + q\mu\delta + u\alpha\lambda + v\lambda\delta\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha\mu + v\alpha\lambda + p\mu\delta\omega + u\lambda\delta\omega\right) \\ \frac{1}{\alpha^2-\delta^2\omega}\left(\alpha\mu - y\delta - x\alpha + \alpha\nu - \lambda\delta + p\alpha\xi + q\delta\xi + u\alpha\nu + v\delta\nu - \delta\xi\omega\right) & -\frac{1}{\alpha^2-\delta^2\omega}(q\alpha\xi - \alpha\lambda - y\alpha + v\alpha\nu - x\delta\omega - \alpha\xi\omega + \mu\delta\omega + \delta\nu\omega \\ -\frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha^2 + q\alpha\delta + v\delta^2\omega + u\alpha\delta\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha^2 + u\delta^2\omega^2 + p\alpha\delta\omega + v\alpha\delta\omega\right) \end{pmatrix}$$

So we need $\lambda = 0$, $\mu = 0$ giving

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ x & y & 1 \\ p & q & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma - \omega\beta\delta) & (\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ (\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & (\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & (\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2-\delta^2\omega}\left(q\delta^2 + u\alpha^2 + p\alpha\delta + v\alpha\delta\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(v\alpha^2 + q\alpha\delta + p\delta^2\omega + u\alpha\delta\omega\right) & 0 \\ 0 & 0 & 0 \\ \frac{1}{\alpha^2-\delta^2\omega}\left(x\alpha + y\delta + \alpha\nu + p\alpha\xi + q\delta\xi + u\alpha\nu + v\delta\nu - \delta\xi\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(y\alpha + q\alpha\xi + v\alpha\nu + x\delta\omega - \alpha\xi\omega + \delta\nu\omega + p\delta\xi\omega + u\delta\nu\omega\right) & 1 \\ \frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha^2 + q\alpha\delta + v\delta^2\omega + u\alpha\delta\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha^2 + u\delta^2\omega^2 + p\alpha\delta\omega + v\alpha\delta\omega\right) & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ x & y & 1 \\ p & q & 0 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma - \omega\beta\delta) & -(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ -(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & -(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & -(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2-\delta^2\omega}\left(q\delta^2 + u\alpha^2 + p\alpha\delta + v\alpha\delta\right) & -\frac{1}{\alpha^2-\delta^2\omega}\left(v\alpha^2 + q\alpha\delta + p\delta^2\omega + u\alpha\delta\omega\right) & 0 \\ 0 & 0 & 0 \\ \frac{1}{\alpha^2-\delta^2\omega}\left(\alpha\nu - y\delta - x\alpha + p\alpha\xi + q\delta\xi + u\alpha\nu + v\delta\nu - \delta\xi\omega\right) & -\frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha\xi - y\alpha + v\alpha\nu - x\delta\omega - \alpha\xi\omega + \delta\nu\omega + p\delta\xi\omega + u\delta\nu\omega\right) & 1 \\ -\frac{1}{\alpha^2-\delta^2\omega}\left(p\alpha^2 + q\alpha\delta + v\delta^2\omega + u\alpha\delta\omega\right) & \frac{1}{\alpha^2-\delta^2\omega}\left(q\alpha^2 + u\delta^2\omega^2 + p\alpha\delta\omega + v\alpha\delta\omega\right) & 0 \end{pmatrix}$$

Note that the values of pa and pd depend only on α, δ (together with their original values), and that replacing α, δ by $\alpha k, \delta k$ makes no difference. There is a MAGMA program to compute the orbits of pa, pd under this action. It isn't particularly easy to see, but for any fixed values of pa, pd, we can always take $x = 0$, and $y = 0$ or 1. Just to make things tricky, for some fixed pa, pd, $x = y = 0$ is in the same orbit as $x = 0$, $y = 1$, and sometimes it isn't. There is a MAGMA program, notes4.1case6.m, to sort this out.