Algebra 6.178

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Algebra 6.178 has presentation

$$\langle a, b, c \,|\, ca - bab,\, cb - \omega baa,\, pa - \lambda baa - \mu bab,\, pb - \nu baa - \xi bab,\, pc,\, \text{class } 3\rangle,$$

where we write $A = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$, and A ranges over a set of representatives for the orbits of non-singular 2×2 matrices under the action

$$A \to \frac{1}{\det P} PAP^{-1}$$

as P ranges over non-singular matrices

$$P = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix}.$$

These algebras are terminal unless $\xi = -\lambda$. The number of orbits of non-singular matrices with $\xi = -\lambda$ is $(3p-1)/2$. The matrices split up into one orbit of size $p-1$ (matrices $\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}$), $p-1$ orbits of size $(p^2-1)/2$ (including two orbits of elements $\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}$), and $(p-1)/2$ orbits of size p^2-1. In all, 6.178 has $(3p^2-1)/2$ descendants of order p^7 and p-class 4. All orbits contain matrices where $\lambda = 0$ or $\lambda = 1$.

It is possible to choose orbit representatives of the following 6 types:

1. $\begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}$,
2. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ when $p = 1 \bmod 4$,
3. $\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$ (all $\begin{pmatrix} 0 & \mu \\ -\omega\mu & 0 \end{pmatrix}$ are in the same orbit as $\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$, but this orbit also contains elements $\begin{pmatrix} 1 & \mu \\ -\omega\mu & -1 \end{pmatrix}$),

4. one representative $\begin{pmatrix} 1 & \mu \\ -\omega\mu & -1 \end{pmatrix}$ ($\mu \neq 0$) which is not in the same orbit as $\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$ when $p = 3 \bmod 4$,

5. $p-3$ representatives $\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}$ ($\nu \neq \pm\omega\mu$), and

6. $(p-1)/2$ representatives of the form $\begin{pmatrix} 1 & \mu \\ \nu & -1 \end{pmatrix}$ ($\nu \neq -\omega\mu$).

We then obtain the following presentations for the descendants of 6.178. The first four cases of the matrix A are straightforward.

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - bab,\ pb - \omega baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (\text{all } x,\ x \sim -x),$$

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - baa,\ pb + bab,\ pc - xbaab,\ \text{class } 4\rangle\ (\text{all } x,\ x \sim -x,\ p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - bab,\ pb + \omega baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (\text{all } x,\ x \sim -x),$$

For the one matrix $\begin{pmatrix} 1 & \mu \\ -\omega\mu & -1 \end{pmatrix}$ ($\mu \neq 0$) when $p = 3 \bmod 4$, we have

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - baa - \mu bab,\ pb + \omega\mu baa + bab,\ pc - xbaaa,\ \text{class } 4\rangle\ (\text{all } x,\ x \sim -x).$$

For the $p-3$ matrices $A = \begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}$ ($\nu \neq \pm\omega\mu$), we have

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - \mu bab,\ pb - \nu baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (\text{all } x,\ x \sim -x),$$

but we have extra descendants if $(\omega\mu + 2\nu)(2\omega\nu + \mu^{-1}\nu^2)$ is a square. If $\omega\mu + 2\nu = 0$ then we have,

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - \mu bab - xbaaa,\ pb - \nu baa,\ pc,\ \text{class } 4\rangle\ (x \neq 0,\ x \sim -x),$$

If $2\omega\nu + \mu^{-1}\nu^2 = 0$ we have

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - \mu bab,\ pb - \nu baa - xbaaa,\ pc,\ \text{class } 4\rangle\ (x \neq 0,\ x \sim -x),$$

and if $(\omega\mu + 2\nu)(2\omega\nu + \mu^{-1}\nu^2) = y^2 \neq 0$ then for one such value y we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}\mu bab,\ pb{-}\nu baa{-}xbaaa,\ pc{-}ybaaa,\ \text{class } 4\rangle\ (x \neq 0,\ x \sim -x).$$

The situation is even more complicated for the $(p-1)/2$ matrices $A = \begin{pmatrix} 1 & \mu \\ \nu & -1 \end{pmatrix}$ $(\nu \neq -\omega\mu)$. First we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}baa{-}\mu bab,\ pb{-}\nu baa{+}bab,\ pc{-}xbaab,\ \text{class } 4\rangle\ (\text{all } x).$$

But if $(1+\mu\nu)\left(2\,(\omega\mu+\nu)^2+\omega(1+\mu\nu)\right)$ is a square we have an additional $p-1$ descendants. It is not that easy to prove, but $(1+\mu\nu)\left(2\,(\omega\mu+\nu)^2+\omega(1+\mu\nu)\right)$ cannot equal zero, under the assumption that A is not in the same orbit as a matrix with $(1,1)$ entry equal to zero. If $(1+\mu\nu)\left(2\,(\omega\mu+\nu)^2+\omega(1+\mu\nu)\right) = x^2 \neq 0$, then if $x-\omega\mu-\nu = \omega\mu^2+2\mu\nu+1 = 0$ we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}baa{-}\mu bab{-}ybaab,\ pb{-}\nu baa{+}bab,\ pc{-}xbaab,\ \text{class } 4\rangle\ (y \neq 0,\ y \sim -y),$$

but if one of $x-\omega\mu-\nu$, $\omega\mu^2+2\mu\nu+1$ is non-zero we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}baa{-}\mu bab,\ pb{-}\nu baa{+}bab{-}ybaab,\ pc{-}xbaab,\ \text{class } 4\rangle\ (y \neq 0,\ y \sim -y).$$

And similarly for $-x$, if $x+\omega\mu+\nu = \omega\mu^2+2\mu\nu+1 = 0$ we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}baa{-}\mu bab{-}ybaab,\ pb{-}\nu baa{+}bab,\ pc{+}xbaab,\ \text{class } 4\rangle\ (y \neq 0,\ y \sim -y),$$

but if one of $x+\omega\mu+\nu$, $\omega\mu^2+2\mu\nu+1$ is non-zero we have

$$\langle a, b, c \,|\, ca{-}bab,\ cb{-}\omega baa,\ pa{-}baa{-}\mu bab,\ pb{-}\nu baa{+}bab{-}ybaab,\ pc{+}xbaab,\ \text{class } 4\rangle\ (y \neq 0,\ y \sim -y).$$

There is a Magma program notes6.178.m which computes a representative set of matrices A for any given p, and then computes representative values for the other parameters for each A.