Algebra 6.173

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Algebra 6.173 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc,\ \text{class } 3\rangle.$$

If L is a descendant of 6.173 of order p^7 then the commutator structure of L is the same as that of one of the $p+2$ algebras with presentations 7.106 and 7.107 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we can assume that L has the following commutator relations

$$ca = bab,\ cb = \omega baa,\ baab = \lambda baaa,\ babb = \mu baaa$$

for some parameters λ, μ.

If we let $C = \langle c\rangle + L^2$ then, if a', b', c' are the images of a, b, c under an automorphism of L, we have

$$\begin{aligned} a' &= \alpha a + \beta b \bmod C, \\ b' &= \pm(\omega\beta a + \alpha b) \bmod C, \\ c' &= (\alpha^2 - \omega\beta^2)c \bmod L^3 \end{aligned}$$

for some α, β which are not both zero. It follows that

$$\begin{aligned} [b', a', a', a'] &= \pm(\alpha^2 - \omega\beta^2)(\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu)[b, a, a, a], \\ [b', a', a', b'] &= (\alpha^2 - \omega\beta^2)(\omega\alpha\beta + \alpha^2\lambda + \omega\beta^2\lambda + \alpha\beta\mu)[b, a, a, a], \\ [b', a', b', b'] &= \pm(\alpha^2 - \omega\beta^2)(\omega^2\beta^2 + 2\omega\alpha\beta\lambda + \alpha^2\mu)[b, a, a, a]. \end{aligned}$$

So provided $\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu \neq 0$ the effect of this automorphism is to transform the parameters λ, μ to

$$\frac{\pm(\omega\alpha\beta + \alpha^2\lambda + \omega\beta^2\lambda + \alpha\beta\mu)}{\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu},\ \frac{\omega^2\beta^2 + 2\omega\alpha\beta\lambda + \alpha^2\mu}{\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu}.$$

There are $p+2$ orbits of pairs λ, μ under this action.

We pick a set representative pairs λ, μ for these orbits, and get the following presentations for the descendants of 6.173 of order p^7:

$\langle a, b, c \mid ca{-}bab,\ cb{-}\omega baa, baab{-}\lambda baaa,\ babb{-}\mu baaa,\ pa{-}ybaaa,\ pb{-}zbaaa,\ pc{-}tbaaa,$ class 4$\rangle$.

For each pair λ, μ we compute the subgroup of the automorphism group which fixes λ, μ, and compute its action on the parameters $y, z, \dot{t}$. It turns out that we need to treat the pair $\lambda = \mu = 0$ separately from the other pairs.

If $\lambda = \mu = 0$. Then the subgroup of the automorphism group we need to consider maps a, b, c to a', b', c' where

$$\begin{aligned} a' &= \alpha a, \\ b' &= \pm\alpha b + \varepsilon c, \\ c' &= \alpha^2 c, \end{aligned}$$

with $b'a'a'a' = \pm\alpha^4 baaa$.

In all other cases we can assume that if $pc \neq 0$ then $pa = pb = 0$. A Magma program to compute a set of representatives for the parameters λ, μ, y, z, t is given in notes6.173.m.