Algebras 6.163 - 6.167

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Algebras 6.163 – 6.167 give a classification of algebras of order p^6 with presentations

$$\langle a,b,c \,|\, ca-baa,\, cb,\, pa-\lambda baa-\mu bab,\, pb+\nu baa+\xi bab,\, pc,\, \text{class } 3\rangle$$

with $\lambda,\mu,\nu,\xi\neq 0$. Most of these algebras are terminal, and we need a slightly different classification of these algebras from that given in the classification of nilpotent Lie rings of order p^6, so as to classify the capable ones. It turns out that $\frac{5}{2}p-\frac{9}{2}+\frac{1}{2}\gcd(p-1,4)$ of these algebras are capable, and that they have a total of $\frac{1}{2}p^3+2p^2-5p+\frac{1}{2}+\frac{p}{2}\gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Let L have the presentation above, and suppose that a',b',c' generate L and satisfy similar relations, but with (possibly) different λ,μ,ν,ξ. Then

$$\begin{aligned}
a' &= \alpha a+\gamma c,\\
b' &= \delta b+\varepsilon c,\\
c' &= \alpha\delta c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
pa' &= \frac{\lambda}{\alpha\delta}b'a'a'+\frac{\mu}{\delta^2}b'a'b',\\
pb' &= \frac{\nu}{\alpha^2}b'a'a'+\frac{\xi}{\alpha\delta}b'a'b'
\end{aligned}$$

or

$$\begin{aligned}
a' &= \alpha b+\gamma c,\\
b' &= \delta a+\varepsilon c,\\
c' &= \alpha\delta c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
pa' &= \frac{\xi}{\alpha\delta}b'a'a'+\frac{\nu}{\delta^2}b'a'b',\\
pb' &= \frac{\mu}{\alpha^2}b'a'a'+\frac{\lambda}{\alpha\delta}b'a'b'.
\end{aligned}$$

So we can take $\lambda = 1$ and $\mu = 1$ or ω (or any other fixed integer which is not a square mod p). Given these values of λ, μ it turns out that the algebra is terminal unless $\xi = 1$ or $\xi = \mu\nu$.

So we have two families of capable algebras of order p^6:

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab,\ pb + \nu baa + bab,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab,\ pb + \nu baa + \mu\nu bab,\ pc, \text{ class } 3\rangle.$$

These two families have immediate descendants of order p^7 with the following presentations involving parameters y, z, t:

$$\langle a, b, c \,|\, ca-baa,\ cb,\ pa-baa-\mu bab-ybaaa,\ pb+\nu baa+bab-zbaaa,\ pc-tbaaa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, ca-baa,\ cb,\ pa-baa-\mu bab-ybaaa,\ pb+\nu baa+\mu\nu bab-zbaaa,\ pc-tbaaa, \text{ class } 3\rangle.$$

For the first family of descendants we consider transformations of the form

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= c, \end{aligned}$$

where if $\mu\nu = 1$ we need $\gamma = \mu\varepsilon$, and transformations of the form

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \alpha^{-1} a + \varepsilon c, \\ c' &= c, \end{aligned}$$

where $\alpha^2\nu = \mu$, and where if $\mu\nu = 1$ we need $\gamma = \mu\varepsilon$. For these transformations we have

$$\begin{aligned} y &\to \pm y + \gamma t + \gamma\mu^{-1} + \varepsilon \\ z &\to \pm z + \varepsilon t - \nu\gamma\mu^{-1} + \nu\varepsilon - 2\varepsilon\mu^{-1}, \\ t &\to t, \end{aligned}$$

and

$$\begin{aligned} y &\to -\alpha z - \gamma t + \varepsilon - \nu\gamma + 2\gamma\mu^{-1}, \\ z &\to -\alpha^{-1} y - \varepsilon t - \gamma\mu^{-1}\nu - \varepsilon\mu^{-1}, \\ t &\to -t - \nu + \mu^{-1}. \end{aligned}$$

For the second family of descendants we can assume that $\mu\nu \neq 1$. We consider transformations of the form

$$\begin{aligned} a' &= \pm a + \mu\varepsilon c, \\ b' &= \pm b + \varepsilon c, \\ c' &= c, \end{aligned}$$

and, when $\mu\nu = -1$ and $p = 1 \bmod 4$, transformations of the form

$$\begin{aligned} a' &= \alpha b + \mu\varepsilon c, \\ b' &= -\alpha^{-1} a + \varepsilon c, \\ c' &= -c, \end{aligned}$$

where $\alpha^2 = -\mu^2$. For these transformations we have

$$\begin{aligned} y &\to \pm y + \mu\varepsilon t + 2\varepsilon, \\ z &\to \pm z + \varepsilon t - 2\nu\varepsilon, \\ t &\to t, \end{aligned}$$

and

$$\begin{aligned} y &\to -\alpha z - \mu\varepsilon t - 2\varepsilon, \\ z &\to \alpha^{-1} y - \varepsilon t + 2\nu\varepsilon, \\ t &\to t. \end{aligned}$$

There is a Magma program to compute representative sets of parameters in notes6.163.m.