Algebra 6.150

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Algebra 6.150 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle$$

and so if L is a descendant of 6.150 of order p^7 then the commutator structure of L is the same as one of 7.108 – 7.115 from the list of nilpotent Lie algbebras over $\mathbb{Z}_p$. So we can assume that one of the following sets of commutator relations holds.

$$\begin{aligned}
baaa &= baab = ca - baa = cb = 0, \\
baaa &= baab = ca - baa - babb = cb = 0, \\
baaa &= baab = babb,\ ca - baa = cb = 0, \\
babb &= -baaa,\ baab = ca - baa = cb = 0, \\
babb &= -\omega baaa,\ baab = ca - baa = cb = 0, \\
baaa &= babb = ca - baa = cb = 0, \\
babb &= baab,\ baaa = ca - baa = cb = 0, \\
baaa &= baab,\ babb = xbaaa,\ ca - baa = cb = 0\ (x \neq 0, 1).
\end{aligned}$$

For each of these cases we obtain a generator d for L_4 (d equals one of $baaa$, $baab$, $babb$) and we write

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = Ad$$

where A is a 3×1 matrix over $\mathbb{Z}_p$. In each case the isomorphism classes of algebras are given by the orbits of the matrices A under a given action by a group of automorphisms. I was able to "solve" the problem in every case, providing presentations with fewer parameters, and explicit relatively simple equivalence relations on the parameter sets. However in four of the cases the equivalence classes were slightly more complex than usual. For example in one case the equivalence classes for a parameter y were $\{\pm y, \pm\frac{\omega}{y}\}$. These four cases were 3,4,5,8, and these are described below. There are MAGMA programs to compute a representative sets of matrices A in these four cases in notes6.150.m.

0.0.1 Case 3

If $baaa = baab = babb$, $ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & -\gamma \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to -\alpha^{-4} \begin{pmatrix} 0 & \alpha & \gamma \\ \alpha & 0 & -\gamma \\ 0 & 0 & \alpha^2 \end{pmatrix} A.$$

0.0.2 Case 4

If $babb + baaa = baab = ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \pm\alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to \alpha^{-4} \begin{pmatrix} 0 & \pm\alpha & \gamma \\ \alpha & 0 & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A.$$

0.0.3 Case 5

If $babb + \omega baaa = baab = ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \pm\alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to \omega^{-2}\alpha^{-4} \begin{pmatrix} 0 & \pm\alpha & \gamma \\ \omega\alpha & 0 & \varepsilon \\ 0 & 0 & \omega\alpha^2 \end{pmatrix} A.$$

0.0.4 Case 8

If $baaa = baab$, $babb = xbaaa$, $ca - baa = cb = 0$ where $x \neq 0, 1$ then L_4 is generated by $baaa$. (If we set $x = 0$ we have Case 7, and if we set $x = 1$ we have Case 3.) The action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to x^{-2}\alpha^{-4} \left(\begin{array}{lll} 0 & \alpha & \gamma \\ x\alpha & 0 & \varepsilon \\ 0 & 0 & -x\alpha^2 \end{array} \right) A.$$