Notes5.14

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1 Immediate descendants of algebra 5.14

Algebra 5.14 has

$$2p^5 + 7p^4 + 19p^3 + 49p^2 + 128p + 256 + (p^2 + 7p + 29)\gcd(p-1,3)$$
$$+(p^2 + 7p + 24)\gcd(p-1,4) + (p+3)\gcd(p-1,5)$$

immediate descendants of order p^7 and p-class 3.

Algebra 5.14 has presentation

$$\langle a, b, c \mid cb,\ pa,\ pb,\ pc,\ \text{class } 2\rangle,$$

and so if L is an immediate descendant of 5.14 of order p^7 then L_2 is generated by ba, ca modulo L_3, and L_3 has order p^2 and is generated by baa, bab, bac, caa, cab. And cb, pa, pb, $pc \in L_3$. The commutator structure is the same as one of 7.65 – 7.88 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$. So we may assume that one of the following holds:

$$cb = caa = cab = cac = 0, \tag{7.65}$$
$$caa = cab = cac = 0,\ cb = baa, \tag{7.66}$$
$$cb = bab = bac = cab = cac = 0, \tag{7.67}$$
$$cb = baa,\ bab = bac = cab = cac = 0, \tag{7.68}$$
$$cb = bac = cac = 0,\ caa = bab, \tag{7.69}$$
$$cb = baa,\ bac = cac = 0,\ caa = bab, \tag{7.70}$$
$$cb = baa = bac = cac = 0, \tag{7.71}$$
$$baa = bac = cac = 0,\ cb = caa, \tag{7.72}$$
$$cb = bac = caa = 0,\ cac = bab, \tag{7.73}$$
$$cb = bac = caa = 0,\ cac = \omega bab, \tag{7.74}$$
$$bac = caa = 0,\ cb = baa,\ cac = bab, \tag{7.75}$$
$$bac = caa = 0,\ cb = baa,\ cac = \omega bab, \tag{7.76}$$
$$cb = bac = 0,\ caa = baa,\ cac = -bab, \tag{7.77}$$
$$bac = 0,\ cb = caa = baa,\ cac = -bab, \tag{7.78}$$
$$cb = baa = bac = caa = 0, \tag{7.79}$$
$$cb = bac = caa = 0,\ baa = cac, \tag{7.80}$$

$$cb = bac = 0,\ baa = cac,\ caa = bab, \tag{7.81}$$

$$cb = bac = 0,\ baa = cac,\ caa = \omega bab,\ (p = 1 \bmod 3) \tag{7.82}$$

$$cb = baa = caa = cac = 0, \tag{7.83}$$

$$cb = baa = cac = 0,\ caa = bab, \tag{7.84}$$

$$cb = caa = cac = 0,\ baa = bab, \tag{7.85}$$

$$cb = baa = caa = 0,\ cac = \omega bab, \tag{7.86}$$

$$cb = baa = 0,\ caa = bac,\ cac = \omega bab, \tag{7.87}$$

$$cb = baa = 0,\ caa = kbab + bac,\ cac = \omega bab,\ (p = 2 \bmod 3), \tag{7.88}$$

where k is any (one) integer which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)} \bmod p.$$

Since the total number of descendants of 5.14 of order p^7 is of order $2p^5$, we need presentations with at least 5 parameters in some of these cases. In each case the commutator structure is determined, and so to give a presentation for the Lie rings we only need to specify pa, pb, pc. These powers take values in L_3, which has order p^2, so we need 2 coefficients for each of pa, pb, pc. For the sake of simplicity I give a single presentation with 6 parameters for each of the 24 different commutator structures defined above, and I give the conditions for two sets of parameters to define isomorphic Lie rings. In most of the cases I was able to "solve" the isomorphism problem in the sense that I was able to produce a number of presentations with fewer parameters, and with simple conditions on the parameters. But I was not able to do this in every case.

The file notes5.14.m gives MAGMA programs to compute each isomorphism class. The programs have complexity at most p^5, in the sense that they have nested loops and the statements in the innermost loops are executed a maximum of $O(p^5)$ times. The programs run reasonably fast for $p < 20$, but you need to take a deep breath before running them for $p > 20$. Apart from anything else the shear number of groups becomes overwhelming pretty quickly. My classification of the nilpotent Lie rings of order p^7 has been criticized on the grounds that the Lie rings for any given prime have to be computed "on the fly". However, as I observed above, you need some presentations with at least 5 parameters, and even if you had five parameters independently taking all values between 0 and $p-1$ you would still need a program of complexity p^5 to print them out.

1.1 Case 1

$$\langle a, b, c \mid cb, caa, cab, cac, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab \rangle.$$

Here L_3 is generated by baa and bab, and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2\lambda}(\alpha x_1+\beta x_3+\gamma x_5) & \frac{1}{\alpha\lambda^2}(\alpha x_2+\beta x_4+\gamma x_6)-\frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\alpha x_1+\beta x_3+\gamma x_5) \\ \frac{1}{\alpha^2\lambda}(\lambda x_3+\mu x_5) & \frac{1}{\alpha\lambda^2}(\lambda x_4+\mu x_6)-\frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\lambda x_3+\mu x_5) \\ \frac{1}{\alpha^2\lambda}\xi x_5 & \frac{1}{\alpha\lambda^2}\xi x_6-\frac{1}{\alpha^2}\frac{\beta}{\lambda^2}\xi x_5 \end{pmatrix}$$

There are $3p+22$ agebras in all in this case.

1.2 Case 2

$\langle a,b,c \,|\, cb-baa, caa, cab, cac, pa-x_1baa-x_2bab, pb-x_3baa-x_4bab, pc-x_5baa-x_6bab\rangle$.

Here L_3 is generated by baa and bab, and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2\lambda}(\alpha x_1+\beta x_3+\gamma x_5) & \frac{1}{\alpha\lambda^2}(\alpha x_2+\beta x_4+\gamma x_6)-\frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\alpha x_1+\beta x_3+\gamma x_5) \\ \frac{1}{\alpha^2\lambda}(\lambda x_3+\mu x_5) & \frac{1}{\alpha\lambda^2}(\lambda x_4+\mu x_6)-\frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\lambda x_3+\mu x_5) \\ \frac{1}{\lambda}x_5 & \frac{\alpha}{\lambda^2}x_6-\frac{\beta}{\lambda^2}x_5 \end{pmatrix}.$$

The total number of algebras in this case is $5p+13+\gcd(p-1,3)+\gcd(p-1,4)$.

1.3 Case 3

$\langle a,b,c \,|\, cb, bab, bac, cab, cac, pa-x_1baa-x_2caa, pb-x_3baa-x_4caa, pc-x_5baa-x_6caa\rangle$.

L_3 is generated by baa and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & \nu & \xi \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu \\ \alpha^2\nu & \alpha^2\xi \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu \\ \alpha^2\nu & \alpha^2\xi \end{pmatrix}^{-1}$$

$$= \frac{1}{\alpha^2\lambda\xi - \alpha^2\mu\nu}$$
$$\times \begin{pmatrix} \alpha\xi x_1 - \alpha\nu x_2 - \beta\nu x_4 + \beta\xi x_3 - \gamma\nu x_6 + \gamma\xi x_5 & \alpha\lambda x_2 - \alpha\mu x_1 + \beta\lambda x_4 - \beta\mu x_3 + \lambda\gamma x_6 - \gamma\mu x_5 \\ \lambda\xi x_3 - \lambda\nu x_4 - \mu\nu x_6 + \mu\xi x_5 & \lambda^2 x_4 - \mu^2 x_5 - \lambda\mu x_3 + \lambda\mu x_6 \\ \xi^2 x_5 - \nu^2 x_4 + \nu\xi x_3 - \nu\xi x_6 & \lambda\nu x_4 - \mu\nu x_3 + \lambda\xi x_6 - \mu\xi x_5 \end{pmatrix}$$

The total number of algebras in this case is $2p + 8 + \gcd(p-1, 4)$.

1.4 Case 4

$\langle a, b, c \mid cb - baa, bab, bac, cab, cac, pa - x_1 baa - x_2 caa, pb - x_3 baa - x_4 caa, pc - x_5 baa - x_6 caa \rangle$.

$$cb = baa,\ bab = bac = cab = cac = 0.$$

L_3 is generated by baa and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & 0 \\ 0 & \nu & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & 0 \\ \alpha^2\nu & \alpha^4 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & 0 \\ 0 & \nu & \alpha^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & 0 \\ \alpha^2\nu & \alpha^4 \end{pmatrix}^{-1}$$

$$= \frac{1}{\alpha^4} \begin{pmatrix} \frac{1}{\lambda}\left(\alpha^3 x_1 + \alpha^2\beta x_3 + \alpha^2\gamma x_5 - \alpha\nu x_2 - \beta\nu x_4 - \gamma\nu x_6\right) & \alpha x_2 + \beta x_4 + \gamma x_6 \\ \alpha^2 x_3 - \nu x_4 & \lambda x_4 \\ \frac{1}{\lambda}\left(\alpha^4 x_5 - \nu^2 x_4 + \alpha^2\nu x_3 - \alpha^2\nu x_6\right) & x_6\alpha^2 + \nu x_4 \end{pmatrix}$$

The total number of algebras in this case is $6p + 8 + 2\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$.

1.5 Case 5

$\langle a, b, c \mid cb, bac, caa - bab, cac, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab \rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^{-1}\lambda^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu + \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^{-1}\lambda^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu + \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}$$

$$= \frac{1}{\alpha^2\lambda^3} \times$$
$$\begin{pmatrix} \lambda^2(\alpha x_1 + \beta x_3 + \gamma x_5) & \alpha^2\lambda x_2 - \alpha^2\mu x_1 - \beta^2\lambda x_3 - \alpha\beta\lambda x_1 + \alpha\beta\lambda x_4 - \alpha\beta\mu x_3 + \alpha\lambda\gamma x_6 - \alpha\gamma\mu x_5 - \beta\lambda\gamma x_5 \\ \lambda^2(\lambda x_3 + \mu x_5) & \alpha\lambda^2 x_4 - \beta\lambda^2 x_3 - \alpha\mu^2 x_5 - \alpha\lambda\mu x_3 + \alpha\lambda\mu x_6 - \beta\lambda\mu x_5 \\ \frac{1}{\alpha}\lambda^4 x_5 & -\frac{1}{\alpha}\lambda^2(\alpha\mu x_5 - \alpha\lambda x_6 + \beta\lambda x_5) \end{pmatrix}$$

The total number of algebras in this case is $5p + 13 + 2\gcd(p-1,3) + \gcd(p-1,4)$.

1.6 Case 6

$\langle a, b, c \mid cb - baa, bac, caa - bab, cac, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab \rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha^2 & \beta & \gamma \\ 0 & \pm\alpha^3 & \mu \\ 0 & 0 & \alpha^4 \end{pmatrix} A \begin{pmatrix} \pm\alpha^7 & \alpha^4\mu \pm \alpha^5\beta \\ 0 & \alpha^8 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha^2 & \beta & \gamma \\ 0 & \alpha^3 & \mu \\ 0 & 0 & \alpha^4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^7 & \alpha^4\mu + \alpha^5\beta \\ 0 & \alpha^8 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \frac{1}{\alpha^7}(x_1\alpha^2 + \beta x_3 + \gamma x_5) & \frac{1}{\alpha^8}(x_2\alpha^2 + \beta x_4 + \gamma x_6) - \frac{1}{\alpha^{11}}(\mu + \alpha\beta)(x_1\alpha^2 + \beta x_3 + \gamma x_5) \\ \frac{1}{\alpha^7}(x_3\alpha^3 + \mu x_5) & \frac{1}{\alpha^8}(x_4\alpha^3 + \mu x_6) - \frac{1}{\alpha^{11}}(\mu + \alpha\beta)(x_3\alpha^3 + \mu x_5) \\ \frac{1}{\alpha^3}x_5 & \frac{1}{\alpha^4}x_6 - \frac{1}{\alpha^7}x_5(\mu + \alpha\beta) \end{pmatrix},$$
$$\begin{pmatrix} \alpha^2 & \beta & \gamma \\ 0 & -\alpha^3 & \mu \\ 0 & 0 & \alpha^4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} -\alpha^7 & \alpha^4\mu - \alpha^5\beta \\ 0 & \alpha^8 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} -\frac{1}{\alpha^7}(x_1\alpha^2 + \beta x_3 + \gamma x_5) & \frac{1}{\alpha^8}(x_2\alpha^2 + \beta x_4 + \gamma x_6) + \frac{1}{\alpha^{11}}(\mu - \alpha\beta)(x_1\alpha^2 + \beta x_3 + \gamma x_5) \\ -\frac{1}{\alpha^7}(\mu x_5 - \alpha^3 x_3) & \frac{1}{\alpha^8}(\mu x_6 - \alpha^3 x_4) + \frac{1}{\alpha^{11}}(\mu - \alpha\beta)(\mu x_5 - \alpha^3 x_3) \\ -\frac{1}{\alpha^3}x_5 & \frac{1}{\alpha^4}x_6 + \frac{1}{\alpha^7}x_5(\mu - \alpha\beta) \end{pmatrix}$$

The total number of algebras in this case is

$$p^2 + 3p - 3 + (p+2)\gcd(p-1,3) + (p+1)\gcd(p-1,4) + (p+1)\gcd(p-1,5).$$

1.7 Case 7

$$\langle a, b, c \,|\, cb, baa, bac, cac, pa - x_1 bab - x_2 caa, pb - x_3 bab - x_4 caa, pc - x_5 bab - x_6 caa \rangle.$$

L_3 is generated by bab and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}.$$

Now

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \frac{1}{\alpha\lambda^2}(\alpha x_1 + \gamma x_5) & \frac{1}{\alpha^2\xi}(\alpha x_2 + \gamma x_6) \\ \frac{1}{\alpha\lambda} x_3 & \frac{1}{\alpha^2}\frac{\lambda}{\xi} x_4 \\ \frac{1}{\alpha\lambda^2}\xi x_5 & \frac{1}{\alpha^2} x_6 \end{pmatrix}.$$

The total number of algebras in this case is $2p^2 + 11p + 43 + \gcd(p-1, 4)$.

1.8 Case 8

$$\langle a, b, c \,|\, cb - caa, baa, bac, cac, pa - x_1 bab - x_2 caa, pb - x_3 bab - x_4 caa, pc - x_5 bab - x_6 caa \rangle.$$

L_3 is generated by bab and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}.$$

Now

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \frac{1}{\alpha^5}(\alpha x_1 + \gamma x_5) & \frac{1}{\alpha^2\xi}(\alpha x_2 + \gamma x_6) \\ \frac{1}{\alpha^3} x_3 & \frac{1}{\xi} x_4 \\ \frac{1}{\alpha^5}\xi x_5 & \frac{1}{\alpha^2} x_6 \end{pmatrix}.$$

The total number of algebras in this case is

$$p^3 + 4p^2 + 6p + (p+5)\gcd(p-1, 3) + 3\gcd(p-1, 4) + \gcd(p-1, 5).$$

1.9 Case 9

$\langle a, b, c \mid cb, bac, caa, cac - bab, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab\rangle$.
L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \pm\lambda \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2\lambda}(\alpha x_1 + \beta x_3) & \frac{1}{\alpha\lambda^2}(\alpha x_2 + \beta x_4) - \frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\alpha x_1 + \beta x_3) \\ \frac{1}{\alpha^2} x_3 & \frac{1}{\alpha\lambda} x_4 - \frac{1}{\alpha^2}\frac{\beta}{\lambda} x_3 \\ \frac{1}{\alpha^2} x_5 & \frac{1}{\alpha\lambda} x_6 - \frac{1}{\alpha^2}\frac{\beta}{\lambda} x_5 \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{\alpha^2\lambda}(\alpha x_1 + \beta x_3) & \frac{1}{\alpha\lambda^2}(\alpha x_2 + \beta x_4) - \frac{1}{\alpha^2}\frac{\beta}{\lambda^2}(\alpha x_1 + \beta x_3) \\ \frac{1}{\alpha^2} x_3 & \frac{1}{\alpha\lambda} x_4 - \frac{1}{\alpha^2}\frac{\beta}{\lambda} x_3 \\ -\frac{1}{\alpha^2} x_5 & \frac{1}{\alpha^2}\frac{\beta}{\lambda} x_5 - \frac{1}{\alpha\lambda} x_6 \end{pmatrix} =$$

The total number of algebras in this case is

$$p^3 + \frac{5}{2}p^2 + 7p + \frac{19}{2} + \frac{p+4}{2}\gcd(p-1, 4).$$

1.10 Case 10

$\langle a, b, c \mid cb, bac, caa, cac - \omega bab, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab\rangle$.
L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \pm\lambda \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

This case is identical to Case 9 and so there are

$$p^3 + \frac{5}{2}p^2 + 7p + \frac{19}{2} + \frac{p+4}{2}\gcd(p-1, 4)$$

algebras here.

1.11 Case 11

$\langle a,b,c \mid cb-baa, bac, caa, cac-bab, pa-x_1baa-x_2bab, pb-x_3baa-x_4bab, pc-x_5baa-x_6bab\rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \pm\alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} \pm\alpha^4 & \pm\alpha^3\beta \\ 0 & \alpha^5 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^4 & \alpha^3\beta \\ 0 & \alpha^5 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^4}(\alpha x_1 + \beta x_3) & \frac{1}{\alpha^5}(\alpha x_2 + \beta x_4) - \frac{1}{\alpha^6}\beta(\alpha x_1 + \beta x_3) \\ \frac{1}{\alpha^2}x_3 & \frac{1}{\alpha^3}x_4 - \frac{1}{\alpha^4}\beta x_3 \\ \frac{1}{\alpha^2}x_5 & \frac{1}{\alpha^3}x_6 - \frac{1}{\alpha^4}\beta x_5 \end{pmatrix},$$

$$\begin{pmatrix} \alpha & \beta & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} -\alpha^4 & -\alpha^3\beta \\ 0 & \alpha^5 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{1}{\alpha^4}(\alpha x_1 + \beta x_3) & \frac{1}{\alpha^5}(\alpha x_2 + \beta x_4) - \frac{1}{\alpha^6}\beta(\alpha x_1 + \beta x_3) \\ \frac{1}{\alpha^2}x_3 & \frac{1}{\alpha^4}\beta x_3 - \frac{1}{\alpha^3}x_4 \\ -\frac{1}{\alpha^2}x_5 & \frac{1}{\alpha^3}x_6 - \frac{1}{\alpha^4}\beta x_5 \end{pmatrix}.$$

The total number of algebras in this case is

$$(p^4 + p^3 + 4p^2 + p - 1 + (p^2 + 2p + 3)\gcd(p-1,3) + (p+2)\gcd(p-1,4))/2$$

1.12 Case 12

$\langle a,b,c \mid cb-baa, bac, caa, cac-\omega bab, pa-x_1baa-x_2bab, pb-x_3baa-x_4bab, pc-x_5baa-x_6bab\rangle$.

This case is identical to Case 11, so again there are

$$(p^4 + p^3 + 4p^2 + p - 1 + (p^2 + 2p + 3)\gcd(p-1,3) + (p+2)\gcd(p-1,4))/2$$

algebras here.

1.13 Case 13

$\langle a,b,c \mid cb, bac, caa-baa, cac+bab, pa-x_1baa-x_2bab, pb-x_3baa-x_4bab, pc-x_5baa-x_6bab\rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \lambda & \mu \\ 0 & \mu & \lambda \end{pmatrix} A \begin{pmatrix} \alpha^2(\lambda+\mu) & \alpha\beta(\lambda+\mu) \\ 0 & \alpha(\lambda^2-\mu^2) \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \lambda & \mu \\ 0 & \mu & \lambda \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} \alpha^2(\lambda+\mu) & \alpha\beta(\lambda+\mu) \\ 0 & \alpha(\lambda^2-\mu^2) \end{pmatrix}^{-1}$$

$$= \frac{1}{\alpha^2\lambda^2-\alpha^2\mu^2} \begin{pmatrix} (\lambda-\mu)(\alpha x_1+\beta x_3-\beta x_5) & \alpha^2x_2-\beta^2x_3+\beta^2x_5-\alpha\beta x_1+\alpha\beta x_4-\alpha\beta x_6 \\ (\lambda-\mu)(\lambda x_3+\mu x_5) & \alpha\lambda x_4-\beta\lambda x_3+\alpha\mu x_6-\beta\mu x_5 \\ (\lambda-\mu)(\mu x_3+\lambda x_5) & \alpha\mu x_4-\beta\mu x_3+\alpha\lambda x_6-\beta\lambda x_5 \end{pmatrix}.$$

In this case there are $2p^2+11p+27+\gcd(p-1,4)$ immediate descendants of order p^7 and p-class 3.

1.14 Case 14

$\langle a,b,c \mid cb{-}baa, bac, caa{-}baa, cac{+}bab, pa{-}x_1baa{-}x_2bab, pb{-}x_3baa{-}x_4bab, pc{-}x_5baa{-}x_6bab\rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \lambda & \lambda-\alpha^2 \\ 0 & \lambda-\alpha^2 & \lambda \end{pmatrix} A \begin{pmatrix} 2\alpha^2\lambda-\alpha^4 & 2\alpha\beta\lambda-\alpha^3\beta \\ 0 & 2\alpha^3\lambda-\alpha^5 \end{pmatrix}^{-1}.$$

In this case there are $p^3+2p^2+6p+10+(p+4)\gcd(p-1,3)$ algebras.

1.15 Case 15

$\langle a,b,c \mid cb, baa, bac, caa, pa-x_1bab-x_2cac, pb-x_3bab-x_4cac, pc-x_5bab-x_6cac\rangle$.

L_3 is generated by bab and cac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ cac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1}$$

and

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} A \begin{pmatrix} 0 & \alpha\beta^2 \\ \alpha\gamma^2 & 0 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\beta^2} & \frac{y}{\gamma^2} \\ \frac{1}{\beta}\frac{z}{\alpha} & \beta\frac{t}{\alpha\gamma^2} \\ \gamma\frac{u}{\alpha\beta^2} & \frac{1}{\gamma}\frac{v}{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & \alpha\beta^2 \\ \alpha\gamma^2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{y}{\beta^2} & \frac{x}{\gamma^2} \\ \frac{1}{\beta}\frac{v}{\alpha} & \beta\frac{u}{\alpha\gamma^2} \\ \gamma\frac{t}{\alpha\beta^2} & \frac{1}{\gamma}\frac{z}{\alpha} \end{pmatrix}$$

The total number of algebras in this case is

$$p^3 + \frac{7}{2}p^2 + \frac{17}{2}p + \frac{59}{2} + \frac{5}{2}\gcd(p-1,3) + \frac{p+1}{2}\gcd(p-1,4)$$

1.16 Case 16

$\langle a, b, c \mid cb, bac, caa, cac - baa, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab \rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \alpha^2\frac{y}{\gamma^4} \\ \frac{1}{\alpha^2}z & \frac{1}{\gamma^2}t \\ \frac{1}{\gamma}\frac{u}{\alpha} & \frac{1}{\gamma^3}v\alpha \end{pmatrix}.$$

The total number of algebras here is

$$2p^4 + 4p^3 + 8p^2 + 14p + 11 + 4\gcd(p-1,3) + 3\gcd(p-1,4).$$

1.17 Case 17

$\langle a, b, c \mid cb, bac, caa - bab, cac - baa, pa - x_1 baa - x_2 bab, pb - x_3 baa - x_4 bab, pc - x_5 baa - x_6 bab \rangle$.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}$$

or

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \alpha^{-1}\gamma^2 \\ 0 & \gamma & 0 \end{pmatrix} A \begin{pmatrix} 0 & \alpha\gamma^2 \\ \alpha^2\gamma & 0 \end{pmatrix}^{-1}$$

with $\alpha^3 = \gamma^3$.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \frac{1}{\alpha}\frac{y}{\gamma} \\ \frac{1}{\alpha^2}z & \frac{1}{\alpha^3}\gamma t \\ \frac{1}{\gamma}\frac{u}{\alpha} & \frac{v}{\alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \alpha^{-1}\gamma^2 \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & \alpha\gamma^2 \\ \alpha^2\gamma & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{y}{\gamma^2} & \frac{1}{\alpha}\frac{x}{\gamma} \\ \frac{v}{\alpha^2} & \frac{1}{\alpha^3}\gamma u \\ \frac{1}{\gamma}\frac{t}{\alpha} & \frac{1}{\alpha^2}z \end{pmatrix}$$

If $p \neq 1 \bmod 3$ then $\alpha = \gamma$ and the number of orbits is

$$p^5 + p^4 + p^3 + p^2 + p + 2 + (p^2 + p + 1)\gcd(p-1, 4)/2.$$

If $p = 1 \bmod 3$ then $\alpha = \gamma$ or $\xi\gamma$ or $\xi^2\gamma$ where $\xi^3 = 1$. The number of orbits is then

$$(p^5 + p^4 + p^3 + p^2 + 7p + 10)/3 + (p^2 + p + 1)\gcd(p-1, 4)/2$$

So in general the number of orbits is

$$(p^4 + 2p^3 + 3p^2 + 4p + 2)\frac{p-1}{\gcd(p-1, 3)} + 3p + 4 + (p^2 + p + 1)\gcd(p-1, 4)/2$$

1.18 Case 18

$\langle a, b, c \mid cb, bac, caa-\omega bab, cac-baa, pa-x_1baa-x_2bab, pb-x_3baa-x_4bab, pc-x_5baa-x_6bab\rangle$ $(p = 1 \bmod 3)$.

This case is very similar to Case 17, though we do not have as many automorphisms. L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}$$

with $\alpha^3 = \gamma^3$.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \frac{1}{\alpha}\frac{y}{\gamma} \\ \frac{1}{\alpha^2}z & \frac{1}{\alpha^3}\gamma t \\ \frac{1}{\gamma}\frac{u}{\alpha} & \frac{v}{\alpha^2} \end{pmatrix}$$

The number of algebras is

$$(2p^5 + 2p^4 + 2p^3 + 2p^2 + 14p + 17)/3$$

Combining Case 17 and Case 18, the total number of algebras in the two cases is

$$p^5 + p^4 + p^3 + p^2 - 2p - \frac{3}{2} + (3p + \frac{7}{2}) \gcd(p-1, 3) + (p^2 + p + 1) \gcd(p-1, 4)/2$$

1.19 Case 19

$$\langle a, b, c \mid cb, baa, caa, cac, pa - x_1 bab - x_2 bac, pb - x_3 bab - x_4 bac, pc - x_5 bab - x_6 bac \rangle.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & 0 & \delta \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 2\alpha\beta\gamma \\ 0 & \alpha\beta\delta \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 2\alpha\beta\gamma \\ 0 & \alpha\beta\delta \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{x}{\beta^2} & \frac{y}{\beta\delta} - 2\frac{x}{\beta^2}\frac{\gamma}{\delta} \\ \frac{1}{\alpha\beta^2}(u\gamma + z\beta) & \frac{1}{\alpha\beta\delta}(t\beta + v\gamma) - \frac{2}{\alpha\beta^2}\frac{\gamma}{\delta}(u\gamma + z\beta) \\ \frac{u}{\alpha\beta^2}\delta & \frac{v}{\alpha\beta} - 2\frac{u}{\alpha\beta^2}\gamma \end{pmatrix}.$$

The total number of algebras in this case is $2p^2 + 11p + 27 + \gcd(p-1, 4)$.

1.20 Case 20

$$\langle a, b, c \mid cb, baa, caa - bab, cac, pa - x_1 bab - x_2 bac, pb - x_3 bab - x_4 bac, pc - x_5 bab - x_6 bac \rangle.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\beta^2} & \alpha\frac{y}{\beta^3} \\ \frac{1}{\beta}\frac{z}{\alpha} & \frac{1}{\beta^2}t \\ \frac{1}{\alpha^2}u & \frac{1}{\alpha\beta}v \end{pmatrix}.$$

The total number of algebras here is

$$2p^4 + 4p^3 + 6p^2 + 11p + 11 + 2\gcd(p-1, 3) + (p+1)\gcd(p-1, 4).$$

1.21 Case 21

$\langle a, b, c \mid cb, bab - baa, caa, cac, pa - x_1 baa - x_2 bac, pb - x_3 baa - x_4 bac, pc - x_5 baa - x_6 bac \rangle.$

L_3 is generated by baa and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 2\beta \\ 0 & \alpha & \beta \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha^3 & 2\alpha^2\beta \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & 0 & 2\beta \\ 0 & \alpha & \beta \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha^3 & 2\alpha^2\beta \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^3}(2u\beta + x\alpha) & \frac{1}{\alpha^2\gamma}(2v\beta + y\alpha) - \frac{2}{\alpha^3}\frac{\beta}{\gamma}(2u\beta + x\alpha) \\ \frac{1}{\alpha^3}(u\beta + z\alpha) & \frac{1}{\alpha^2\gamma}(t\alpha + v\beta) - \frac{2}{\alpha^3}\frac{\beta}{\gamma}(u\beta + z\alpha) \\ \frac{u}{\alpha^3}\gamma & \frac{v}{\alpha^2} - 2\frac{u}{\alpha^3}\beta \end{pmatrix}.$$

The total number of algebras in this case is

$$2p^3 + 6p^2 + 7p + 7 + (p+1)\gcd(p-1, 4).$$

1.22 Case 22

$\langle a, b, c \mid cb, baa, caa, cac - \omega bab, pa - x_1 bab - x_2 bac, pb - x_3 bab - x_4 bac, pc - x_5 bab - x_6 bac \rangle.$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \omega\beta & \pm\gamma \\ 0 & \omega\gamma & \pm\omega\beta \end{pmatrix} A \begin{pmatrix} \omega\alpha(\omega\beta^2 + \gamma^2) & \pm 2\omega\alpha\beta\gamma \\ 2\omega^2\alpha\beta\gamma & \pm\omega\alpha(\omega\beta^2 + \gamma^2) \end{pmatrix}^{-1}.$$

The total number of algebras in Case 22 is

$$(2p^3 + 3p^2 + 3p + 13 - \gcd(p-1, 3) + (p+1)\gcd(p-1, 4))/2.$$

1.23 Case 23

$\langle a, b, c \mid cb, baa, caa - bac, cac - \omega bab, pa - x_1 bab - x_2 bac, pb - x_3 bab - x_4 bac, pc - x_5 bab - x_6 bac \rangle$.

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \pm\alpha \end{pmatrix} A \begin{pmatrix} \alpha^3 & 0 \\ 0 & \pm\alpha^3 \end{pmatrix}^{-1}$$

or when $p = 2 \bmod 3$ and $12\omega\beta^2 = -1$,

$$A \to \begin{pmatrix} 4\omega\alpha\beta & -3\omega\alpha\beta & \frac{\alpha}{2} \\ 0 & -2\omega\alpha\beta & \alpha \\ 0 & \pm\omega\alpha & \mp 2\omega\alpha\beta \end{pmatrix} A \begin{pmatrix} \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega\alpha^3 \\ \pm\frac{4}{3}\omega^2\alpha^3 & \pm\frac{8}{3}\omega^2\alpha^3\beta \end{pmatrix}^{-1}.$$

Now

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \pm\alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha^3 & 0 \\ 0 & \pm\alpha^3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\alpha^2}x & \pm\frac{1}{\alpha^2}y \\ \frac{1}{\alpha^2}z & \pm\frac{1}{\alpha^2}t \\ \pm\frac{1}{\alpha^2}u & \frac{1}{\alpha^2}v \end{pmatrix}$$

and so if $p = 1 \bmod 3$ there are $p^5 + p^4 + p^3 + p^2 + p + 2 + (p^2 + p + 1)\gcd(p-1, 4)/2$ algebras.

When $p = 2 \bmod 3$ the number of algebras here is

$$\frac{1}{3}p^5 + \frac{1}{3}p^4 + \frac{1}{3}p^3 + \frac{1}{3}p^2 + p + 2 + (p^2 + p + 1)\gcd(p-1, 4)/2.$$

1.24 Case 24

$\langle a, b, c \mid cb, baa, caa - kbab - bac, cac - \omega bab, pa - x_1 bab - x_2 bac, pb - x_3 bab - x_4 bac, pc - x_5 bab - x_6 bac \rangle$ $(p = 2 \bmod 3)$.

where k is any (fixed) integer which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)} \bmod p.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} A \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^3 \end{pmatrix}^{-1}$$

and

$$A \to \left(\begin{array}{lll} -4\alpha & k\alpha\beta + 3\alpha & 3k\omega^{-1}\alpha + \alpha\beta \\ 0 & 2\alpha & 2\alpha\beta \\ 0 & 2\omega\alpha\beta & 2\alpha \end{array} \right) A \left(\begin{array}{ll} 32\alpha^3 & -32\alpha^3\beta \\ -32\omega\alpha^3\beta & 32\alpha^3 \end{array} \right)^{-1}$$

where $\omega\beta^2 = -3$.

The number of orbits is

$$\frac{2}{3}p^5 + \frac{2}{3}p^4 + \frac{2}{3}p^3 + \frac{2}{3}p^2 + 2p + 3.$$

The total number of algebras from Case 23 and Case 24 is

$$p^5 + p^4 + p^3 + p^2 + 4p + \frac{13}{2} - (p + \frac{3}{2}) \gcd(p-1,3) + (p^2 + p + 1) \gcd(p-1,4)/2.$$

The total number of algebras from cases 17, 18, 23 and 24 is

$$p^5 + p^4 + p^3 + p^2 + 2p + 5 + (2p + 2) \gcd(p-1,3) + (p^2 + p + 1) \gcd(p-1,4).$$