Algebra 5.38

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Algebra 5.38 has $\gcd(p-1,3)(p^2+3p+10)+p+6$ immediate descendants of order p^7. Of these $\frac{1}{2}((p^2+3p+11)\gcd(p-1,3)+1)$ come from one 4 parameter family of algebras, and $\frac{1}{2}(\gcd(p-1,3)(p^2+p+1)+5)$ come from another four parameter family. In both cases we take the four parameters as entries in a 2×2 matrix

$$A=\begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

and in both cases we consider the orbits of matrices A of this form over $\mathrm{GF}(p)$ under an action of the subgroup of $\mathrm{GL}(2,p)$ consisting of non-singular matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}.$$

In the first case two matrices A and B give isomorphic Lie rings if and only if

$$B=\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} A \begin{pmatrix} (\alpha^4-\beta^4) & 2\alpha\beta(\alpha^2-\beta^2) \\ 2\alpha\beta(\alpha^2-\beta^2) & \alpha^4-\beta^4 \end{pmatrix}^{-1}$$

or

$$B=\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} A \begin{pmatrix} -(\alpha^4-\beta^4) & -2\alpha\beta(\alpha^2-\beta^2) \\ 2\alpha\beta(\alpha^2-\beta^2) & \alpha^4-\beta^4 \end{pmatrix}^{-1}$$

for some α,β. In the second case, two matrices A and B give isomorphic Lie rings if and only if

$$B=\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} A \begin{pmatrix} \alpha^4-\omega^2\beta^4 & 2\alpha\beta(\alpha^2-\omega\beta^2) \\ 2\omega\alpha\beta(\alpha^2-\omega\beta^2) & \alpha^4-\omega^2\beta^4 \end{pmatrix}^{-1}$$

or

$$B=\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} A \begin{pmatrix} -(\alpha^4-\omega^2\beta^4) & -2\alpha\beta(\alpha^2-\omega\beta^2) \\ 2\omega\alpha\beta(\alpha^2-\omega\beta^2) & \alpha^4-\omega^2\beta^4 \end{pmatrix}^{-1}$$

for some α,β.

A simple loop over all possible A and all possible α,β can find representatives for the orbits. You can shorten the search slightly by noting that in both cases if we take $\alpha=-1$, $\beta=0$ then $B=-A$.